Math2050A Term1 2016
Tutorial 1, Sept 15

Exercises

1. Let $S = [a, b)$, where $a < b$. Find $inf(S)$ and $sup(S)$.

2. Let $S = \{\frac{n}{2^n} : n \in \mathbb{N}\}$ ($0 \notin \mathbb{N}$ in our definition). Find $inf(S)$ and $sup(S)$.

3. (p.45 Q4 in our textbook)
   Let $A \subset \mathbb{R}$ such that $A \neq \emptyset$ and bounded above. Let $b > 0$. Define $bA = \{ba : a \in A\}$. Show that $sup(bA) = bsup(A)$. What is $sup(bA)$ if $b < 0$ and A is bounded below?

4. (p.45 Q7 in our textbook)
   let $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, be nonempty sets. Show that $sup(A + B) = sup(A) + sup(B)$ whenever A,B are bounded above and $inf(A+B) = inf(A) + inf(B)$ whenever A, B are bounded below.

5. (p.45 Q11 in our textbook)
   Let X, Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have a bounded range. Define

   $$f(x) = sup\{h(x, y) : y \in Y\}, g(y) = inf\{h(x, y) : x \in X\}$$

   Show that $sup\{g(y) : y \in Y\} \leq inf\{f(x) : x \in X\}$

Solution (Outline)

1. I only find $sup(S)$ here. We claim that $sup(S) = b$
   Proof:
   For $s \in S$, $s < b$. Therefore, b is an upper bound.
   We want to show that if u is an upper bound of S, then $u \geq b$:
   Since u is an upper bound of S, $u \geq s \ \forall s \in S = [a, b)$.
   This implies also $u \geq s$ whenever $b > s$. This says that $u \geq b$. Let's see:
   Suppose $u < b$, take $\alpha = \frac{u+b}{2}$, then $b > \alpha$ but $u < \alpha$. Contradicts to "$u \geq s$ whenever $b > s$ ". Therefore, $u \geq b$.
   By definition of supremum, $sup(S) = b$
   You can also argue that "if $v < b$, then v cannot be an upper bound of S" to conclude that b is the least, as what we do in Q2.

2. Since $\frac{n+1}{2^{n+1}} = \frac{1}{2}(\frac{n}{2^n}) + \frac{1}{2}(\frac{1}{2^n})$ is taking mean value of $\frac{n}{2^n}$ and $\frac{1}{2^n}$,
   we have $\frac{n+1}{2^{n+1}} \leq \frac{n}{2^n}$.
   Therefore, $\frac{1}{2}$ is the maximum of S. We have $sup(S) = \frac{1}{2}$
   For the infimum, we claim that $inf(S) = 0$:
   0 is obviously a lower bound of S.
   We claim that any postive number cannot be a lower bound of S:
   Let $\epsilon > 0$ be any postive number.
   Since $2^n = (1+1)^n = 1+n+\frac{n(n-1)}{2}+... \geq \frac{n(n-1)}{2} \ \forall n \geq 2$, then $\frac{n}{2^n} \leq \frac{2}{n-1}$.
   By Archimedean property, there is $N \in \mathbb{N}$ such that $1/N < \epsilon/2$. Take $n = N + 1$, we have $\frac{n}{2^n} \leq \frac{2}{n-1} = \frac{2}{N} < \epsilon$. Hence ϵ is not a lower bound of S.

3. bA is a nonempty subset of $\mathbb{R}$ bounded above. Hence, $sup(bA)$ exists. $sup(bA) \geq ba \ \forall a \in A$. Then, $\frac{1}{b}sup(bA) \geq a \ \forall a \in A$. Since LHS is a constant and upper bound of A, we have $\frac{1}{b}sup(bA) \geq sup(A)$, hence $sup(bA) \geq bsup(A)$.
   One can conclude that for any nonempty subset of $\mathbb{R}$ bounded above, say B, and any postive number, say c, we have $sup(cB) \geq csup(B)$. Put $c = \frac{1}{b}$ and $B = bA$, one can obtain the other inequality sign. (Check $\frac{1}{b}bA = A$)

4. I only do the following: Assume $sup(A+B) = sup(A) + sup(B)$ whenever A,B is bounded above, we show that $inf(A + B) = inf(A) + inf(B)$ whenever A,B are bounded below. Define $-A = \{-a : a \in A\}$, then by assumption, $sup((-A) + (-B)) = sup(-A) + sup(-B)$ and hence $-inf(A+B) = -inf(A) - inf(B)$. You need to check that $-A$ is bounded above, $sup(-A) = -inf(A)$ and $-(A+B) = (-A)+(-B)$

5. Since $\{h(x, y) : y \in Y\}$ and $\{h(x, y) : x \in X\}$ are nonempty bounded subset of $\mathbb{R}$, respectively for each $x \in X$ and $y \in Y$, then both $f(x)$ and $g(y)$ are well-defined.
   For each $y \in Y$, $g(y) \leq h(x, y)$ $\forall x \in X$. For each $x \in X$, $h(x, y) \leq f(x)$ $\forall y \in Y$. Now, let $x_0 \in X$ being fixed, then for each $y \in Y$, $g(y) \leq h(x_0, y) \leq f(x_0)$. Since $f(x_0)$ is a constant and an upper bound of $\{g(y) : y \in Y\}$, we have $sup\{g(y) : y \in Y\} \leq f(x_0)$. This holds for all x_0 in X and LHS is a constant as well as a lower bound of $\{f(x) : x \in X\}$, so $sup\{g(y) : y \in Y\} \leq inf\{f(x) : x \in X\}$.